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ON A GENERALIZATION OF SOME FORMULAS OF THE THEORY OF "MODERATELY THICK" ELASTIC PLATES†

ERIC REISSNER
Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, La Jolla, CA 92093, U.S.A.

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Abstract—We first restate formulas for stresses and stress couples of a theory of *isotropic* moderately thick plates in the classical texts of Love and of Timoshenko in simplified form. We then use recent results for a two-dimensional twelfth-order plate theory, to derive a generalization of the classical formulas such that the results for the isotropic case appear as special cases of the corresponding results for *transversely* isotropic plates. In addition we transform the general equations of our two-dimensional twelfth-order theory so as to be left with one fourth-order equation for an interior solution contribution, and one fourth-order equation for transverse *normal* stress edge zone effects, in addition to the previously known two simultaneous second-order equations for transverse *shear* stress edge zone effects.

1. INTRODUCTION

Given the existence of a classical system of exact formulas for stress couples and the associated stress components, for edge loaded "moderately thick" *isotropic* plates[1, 2], we are here concerned with related matters, as follows.

(1) We show that it is possible to write the classical formulas in a simpler, more symmetrical form.

(2) We show that the consequences of a recently derived twelfth-order two-dimensional theory of *transversely* isotropic plates[4] are consistent with the classical exact results in such a way that the classical results for isotropic plates become special cases of results for the case of transverse isotropy.

(3) We show that our generalization of the classical results for moderately thick plates is associated with the *interior* solution portion of the twelfth-order system of two-dimensional differential equations in Ref. [4].

(4) We complement our earlier result of a system of two simultaneous second-order differential equations for the part of the *edge zone* solution contribution which accounts for the effect of transverse *shearing* strain by the derivation of a new fourth-order equation for the part of the edge zone solution contribution which accounts for the effect of transverse *normal* strain.

2. EXPRESSIONS FOR STRESS COUPLES AND STRESS COMPONENTS FOR MODERATELY THICK ISOTROPIC PLATES

With x, y as coordinates in the mid-plane of a plate of thickness $2c$, and with $w_m(x, y)$ as the deflection function for this midplane, the classical formulas for the stress couples in an isotropic plate are, with some slight change of notation, as follows[1, p. 473; 2, pp. 98–104]

$$M_x = -D_M\left[w_{m,xx} + \nu w_{m,yy} - \frac{8+\nu}{10}c^2\nabla^2 w_{m,yy}\right] \tag{1a}$$

† Dedicated to my friend Nicholas Hoff, on the occasion of his eightieth anniversary.

$$M_y = -D_M\left[w_{m,yy} + \nu w_{m,xx} - \frac{8+\nu}{10}c^2\nabla^2 w_{m,xx}\right] \tag{1b}$$

$$M_{zy} = -D_M\left[(1-\nu)w_{m,xy} + \frac{8+\nu}{10}c^2\nabla^2 w_{m,xy}\right]. \tag{1c}$$

The corresponding formulas for stresses are

$$\sigma_z = \frac{-Ez}{1-\nu^2}\left[w_{m,xx} + \nu w_{m,yy} - \left(1 - \frac{2-\nu}{6}\frac{z^2}{c^2}\right)c^2\nabla^2 w_{m,yy}\right] \tag{2a}$$

$$\sigma_y = \frac{-Ez}{1-\nu^2}\left[w_{m,yy} + \nu w_{m,xx} - \left(1 - \frac{2-\nu}{6}\frac{z^2}{c^2}\right)c^2\nabla^2 w_{m,xx}\right] \tag{2b}$$

$$\tau = \frac{-Ez}{1-\nu^2}\left[(1-\nu)w_{m,xy} + \left(1 - \frac{2-\nu}{6}\frac{z^2}{c^2}\right)c^2\nabla^2 w_{m,xy}\right]. \tag{2c}$$

In this E and ν are Young's modulus and Poisson's ratio, and $D_M = 2Ec^3/3(1-\nu^2)$.

Equations (1) and (2) are exact consequences of the equations of three-dimensional elasticity, subject to a stipulation of absent tractions over the two face planes $z = \pm c$, with the deflection function w_m having to satisfy the differential equation $\nabla^2\nabla^2 w_m = 0$.†

It has previously been indicated in Ref. [3] that upon making use of the form of the differential equation for w_m, and upon introducing a function

$$w_m^* = w_m + \frac{8+\nu}{10}\frac{c^2}{1-\nu}\nabla^2 w_m \tag{3}$$

it is possible to write eqns (1a)–(1c) in the simpler symmetrical form

$$(M_z, M_y, M_{zy}) = -D_M\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2},\ (1-\nu)\frac{\partial^2}{\partial x\partial y}\right)w_m^* \tag{4}$$

and we now note that it is correspondingly possible to write eqns (2a)–(2c) in the simpler form

$$(\sigma_x, \sigma_y, \tau) = \frac{-Ez}{1-\nu^2}\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2},\ (1-\nu)\frac{\partial^2}{\partial x\partial y}\right)W \tag{5}$$

where

$$W = w_m + \left(1 - \frac{2-\nu}{6}\frac{z^2}{c^2}\right)\frac{c^2}{1-\nu}\nabla^2 w_m. \tag{6}$$

† We may note that in the context of contemporary asymptotic considerations of interior and edge zone portions of the solution of the three-dimensional problem, as first considered by Friedrichs and Goldenweiser and most recently by Gregory and Wan[5], these classical exact solutions, which are associated with the names of Maurice Levy, Saint Venant and J. H. Michell, are in fact the interior solution portions for the problem of the three-dimensionally isotropic plate acted upon by edge loads only.

3. STRESS COUPLES AND DIFFERENTIAL EQUATIONS OF THE TWELFTH-ORDER TWO-DIMENSIONAL THEORY

The relevant equations obtained in Ref. [4] are as follows†

$$M_x = -D_M[w^*_{,xx} + \nu w^*_{,yy} + (1+\nu)B_{MT}T] \tag{7a}$$

$$M_y = -D_M[w^*_{,yy} + \nu w^*_{,xx} + (1+\nu)B_{MT}T] \tag{7b}$$

$$M_{xy} = -(1-\nu)D_M w^*_{,xy} \tag{7c}$$

$$\begin{aligned} w^* = w + B_Q D_M\{\nabla^2 w + [(1+\nu)B_{MT} - B_{QS}]T\} \\ + B_{QS}D_P\{\nabla^2 v + [(1+\nu)B_{PT} - B_S]T\} \end{aligned} \tag{8}$$

$$\begin{aligned} [C_T - (1+\nu)D_M B_{MT}(2B_{MT} - B_{QS}) - (1+\nu)D_P B_{PT}(2B_{PT} - B_S)]T \\ = v + (1+\nu)D_M B_{MT}\nabla^2 w + (1+\nu)D_P B_{PT}\nabla^2 v \end{aligned} \tag{9}$$

$$D_M\nabla^2\{\nabla^2 w + [(1+\nu)B_{MT} - B_{QS}]T\} = 0 \tag{10}$$

$$D_P\nabla^2\{\nabla^2 v + [(1+\nu)B_{PT} - B_S]T\} = -T. \tag{11}$$

In this w and v are weighted averages of the three-dimensional transverse displacement component u_z, of the form

$$w = \frac{3}{4c}\int_{-c}^{c}\left(1 - \frac{z^2}{c^2}\right)u_z\,dz, \qquad v = \frac{1}{8c}\int_{-c}^{c}\left(1 - 6\frac{z^2}{c^2} + 5\frac{z^4}{c^4}\right)u_z\,dz \tag{12}$$

and the coefficients D, B and C are given in terms of elastic moduli E, E_z, ν, ν_z and G, in the form

$$D_M = \frac{2Ec^3}{3(1-\nu^2)}, \qquad D_P = \frac{7Ec^3}{2(1-\nu^2)}, \qquad C_T = \frac{4c}{3465E_z} \tag{13a}$$

$$B_Q = \frac{3}{5Gc}, \qquad B_{QS} = \frac{1}{35Gc}, \qquad B_S = \frac{4}{315Gc} \tag{13b}$$

$$B_{MT} = \frac{\nu_z}{35\sqrt{(EE_z)}c}, \qquad B_{PT} = \frac{4\nu_z}{315\sqrt{(EE_z)}c} \tag{13c}$$

with $\nu_z/\sqrt{(EE_z)}$ taking the place of a coefficient ν_z/E_z and with the factor 3465 in eqns (13a) replacing a factor 1155 in the corresponding expression in Ref. [4].

In extension of what has been done in Ref. [4] we now observe that eqn (9) may be transformed into an equation for T only, by taking the Bi-Laplacian of eqn (9), and upon then using eqns (10) and (11) in order to eliminate w and v. The resulting equation for T is

$$D_P[C_T - (1-\nu^2)(D_M B_{MT}^2 + D_P B_{PT}^2)]\nabla^4 T - D_P[B_S - 2(1+\nu)B_{PT}]\nabla^2 T + T = 0. \tag{14a}$$

† Upon omitting terms involving the surface load intensity $q(x,y)$, as well as the edge zone solution contribution functions F and K. We note that the formulas in Ref. [4] should be corrected by changing B_{MT}, B_{PT}, B_{Mq}, B_{Pq} into $-B_{MT}$, $-B_{PT}$, $-B_{Mq}$, $-B_{Pq}$, with this affecting eqns (7)–(11), and that eqn (9) has factors $(1+\nu)D_M$ and $(1+\nu)D_P$, which correct factors D_M and D_P in the corresponding formula in Ref. [4].

With the defining relations in eqns (13a)–(13c) this equation can be rewritten as

$$\frac{2c^4}{495}\frac{1-121\nu_z^2/126}{1-\nu^2}\frac{E}{E_z}\nabla^4 T-\frac{4c^2}{45}\frac{1}{1-\nu}\left[\frac{E}{2(1+\nu)G}-\nu_z\sqrt{\left(\frac{E}{E_z}\right)}\right]\nabla^2 T+T=0. \quad (14b)$$

With T determined from eqn (14a) we then have w from eqn (10).

With eqns (14a) and (10) being two sequential fourth-order equations, and with eqns (9)–(11) being altogether an eighth-order system we do not now determine v from the remaining eqn (11). Instead we observe, on the basis of the form of eqns (9), (10) and (14a), that v will come out to be of the form

$$v=-(1+\nu)D_M B_{MT}\nabla^2 w+A_0 T+A_1\nabla^2 T. \quad (15)$$

In this the coefficients A_0 and A_1 follow, in terms of the coefficients in eqns (9)–(11), upon substituting eqn (15) in eqn (9) and upon observing eqns (10) and (14a), from the solution of two simultaneous linear algebraic equations. We do not here carry out this determination of A_0 and A_1, for the following reason. It is evident from eqn (14b) that the function T represents an edge zone solution contribution, which will accordingly not be used within the present context. A similar consideration holds for the determination of w.

4. INTERIOR SOLUTION PORTIONS FOR THE TWELFTH-ORDER TWO-DIMENSIONAL PROBLEM

Given eqns (10), (14a), and (15) for the determination of T, w and v, the interior solution portion of this system comes out to be of the remarkably simple form

$$T_i=0, \qquad \nabla^4 w_i=0, \qquad v_i=-(1+\nu)D_M B_{MT}\nabla^2 w_i. \quad (16)$$

With eqns (16) it follows then from eqn (8) that

$$w_i^*=w_i+D_M B_Q\nabla^2 w_i \quad (17)$$

and from eqns (7a)–(7c) that

$$(M_x,M_y,M_{xy})_i=-D_M\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial y^2}+\nu\frac{\partial^2}{\partial x^2},\ (1-\nu)\frac{\partial^2}{\partial x\partial y}\right)w_i^*. \quad (18)$$

For a comparison with solution (4) of the theory of "moderately thick" plates we observe that we have, on the basis of eqns (12a) and (12b)

$$D_M B_Q=\frac{2}{5}\frac{E}{G}\frac{c^2}{1-\nu^2} \quad (19)$$

and, when $E=2(1+\nu)G$

$$(D_M B_Q)_{\text{isotropic}}=\frac{4}{5}\frac{c^2}{1-\nu}. \quad (20)$$

Remarkably, the factor "4" in this corresponds to a factor "$4+\nu/2$" in eqn (3). We now show that this discrepancy is apparent rather than real, and caused by the distinction between w_m in eqn (3) and w_i in eqn (17). For a verification of this fact we make use of the defining relations (12) for w and v, in conjunction with the stipulation that u_z in eqn (20) is appropriately approximated by an expression of the form

$$u_z = w_m(x, y) + \frac{z^2}{c^2} v_m(x, y). \tag{21}$$

With this we have from (12)

$$w = w_m + \frac{1}{5} v_m, \qquad v = -\frac{4}{105} v_m \tag{22}$$

and therewith an expression for w_m in terms of w and v

$$w_m = w + \frac{21}{4} v \tag{23}$$

and for w_i in terms of w_{im}, on the basis of (16)

$$w_i = w_{im} + \frac{21}{4}(1 + \nu) D_M B_{MT} \nabla^2 w_i. \tag{24}$$

Since $\nabla^2 \nabla^2 w_i = 0$ we further have

$$\nabla^2 w_i = \nabla^2 w_{im} \tag{25}$$

and therewith

$$w_i = w_{im} + \frac{21}{4}(1 + \nu) D_M B_{MT} \nabla^2 w_{im}. \tag{26}$$

The introduction of eqn (26) into eqn (17) gives

$$\begin{aligned} w_i^* &= w_{im} + \left[B_Q D_M + \frac{21}{4}(1 + \nu) B_{MT} D_M \right] \nabla^2 w_{im} \\ &= w_{im} + \frac{1}{10}\left[\frac{4E}{(1 + \nu)G} + \nu_z \sqrt{\left(\frac{E}{E_z}\right)} \right] \frac{c^2}{1 - \nu} \nabla^2 w_{im}. \end{aligned} \tag{27}$$

A comparison of this expression for w_i^* in conjunction with eqn (18), with the expression for w_m^* in eqns (3) and (4) indicates the following. Equations (18) and (27) represent a *generalization* of the results in eqns (3) and (4). The two sets of results coincide with each other upon specializing eqn (27) by setting $\nu_z = \nu$, $E_z = E$ and $E = 2(1 + \nu)G$. In addition to the fact that eqn (27) is a generalization of eqn (3), this equation also provides information on the nature of the two terms in the classical factor $8 + \nu$ in eqn (3). Evidently, the first term represents the effect of transverse shearing strain, while the second much smaller term represents the effect of transverse normal strain.

5. EXPRESSIONS FOR STRESS IN ACCORDANCE WITH THE TWELFTH-ORDER TWO-DIMENSIONAL THEORY

The expression for the bending stress σ_x in Ref. [4] can be written in the form

$$\sigma_x = \frac{3}{2c^2}\left[M_x + \left(1 - \frac{5z^2}{3c^2}\right)P_x\right]\frac{z}{c} \tag{28}$$

and corresponding expressions hold for σ_y and τ_{zy}. In this M_z is given by eqn (7a) and the corresponding expression for P_x in Ref. [4] is

$$P_x = -D_P[v^*_{,xx} + \nu v^*_{,yy} + (1+\nu)B_{PT}T] \tag{29}$$

where

$$\begin{aligned} v^* = v &+ B_S D_P\{\nabla^2 v - [B_S - (1+\nu)B_{PT}]T\} \\ &+ B_{QS}D_M\{\nabla^2 w - [B_{QS} - (1+\nu)B_{MT}]T\}. \end{aligned} \tag{30}$$

Inasmuch as we are limiting attention to interior solution contributions we deduce from this and from eqn (16)

$$v_i^* = v_i + B_{QS}D_M\nabla^2 w_i = D_M[B_{QS} - (1+\nu)B_{MT}]\nabla^2 w_i. \tag{31}$$

The introduction of P_{xi} from eqns (29) and (31) and of M_{xi} from eqns (18) and (27) then gives

$$\begin{aligned} \sigma_x = \frac{-3}{2c^2}\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\Bigg\{&D_M w_{im} + D_M^2\left[B_Q + \frac{21}{4}(1+\nu)B_{MT}\right]\nabla^2 w_{im} \\ &+ D_M D_P\left(1 - \frac{5}{3}\frac{z^2}{c^2}\right)[B_{QS} - (1+\nu)B_{MT}]\nabla^2 w_{im}\Bigg\}\frac{z}{c}. \end{aligned} \tag{32}$$

With the defining relations in eqns (12a)–(12c) we obtain from this, after some rearrangements and cancellations

$$\begin{aligned} \sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\Bigg\{&w_{im} + \Bigg[\frac{E}{2(1+\nu)G} \\ &- \frac{1}{6}\left(\frac{E}{(1+\nu)G} - \nu_z\sqrt{\left(\frac{E}{E_z}\right)}\right)\frac{z^2}{c^2}\Bigg]\frac{c^2}{1-\nu}\nabla^2 w_{im}\Bigg\} \end{aligned} \tag{33}$$

with corresponding expressions for σ_y and τ_{zy}.

The content of eqn (33) is consistent with the contents of eqns (5) and (6), inasmuch as we obtain the first relation in eqns (5) and (6) from eqn (33) upon identifying w_{im} with the quantity w_m in eqn (6) and upon introducing the assumption of isotropy by setting $E = 2(1+\nu)G$, $\nu_z = \nu$ and $E = E_z$. In addition, we now recognize the way in which the terms with $\nabla^2 w_m$ in eqn (6) depend in part on the effect of the transverse shearing strains and in part on the effect of the transverse normal strain.

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